



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

College Alg. p. 492) that $e^{xi} = \cos x + i \sin x$, (i being used for the square root of -1). Since the trigonometric functions of $x + 2a\pi$ are the same as for x (a being any integer) it follows that $e^{(x+2a\pi)i} = \cos(x+2a\pi) + i \sin(x+2a\pi) \dots (1)$. Making $x = \pi$, we have $e^{\pi(1+2a)i} = -1$. $\therefore \log(-1) = \pi(1+2a)i \dots (2)$.

If in equation (1) we make $x=0$ there results $e^{(2a\pi)i} = 1$.

$\therefore \log 1 = (2a\pi)i \dots (3)$. If now we put $u = e^x$, then the ordinary $\log u = x$. But $-u = e^x \cdot (-1) = e^{x+\pi(1+2a)i}$.

Whence, the general $\log(-u) = x + \pi(1+2a)i = \text{ordinary } \log u + \pi(1+2a)i = \log u + \log(-1)$.

Thus it is shown that the constant in question is the impossible expression, $\pi(1+2a)i$, which by equation (2) is $\log(-1)$.

Again $u = e^x = e^{x \cdot (1)} = e^{x+(2a\pi)i}$.

$\therefore$ It may be shown that the *general* $\log u = \text{ordinary } \log u + (2a\pi)i$.

Hence, as a can have any integral value, it appears that any number, either positive or negative, can have an infinite number of logarithms; but only positive numbers can have their logarithms expressed by arithmetical numbers, and then only when $a=0$.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph.D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

CHAPTER FIRST.

[Continued from the March Number.]

EUCLIDEANS.

John Wallis(1616—1703) deduces Euclid's Parallel-Postulate from the assumption of two similar figures, for example, two unequal triangles having their angles respectively equal and their sides proportional.

But we know that there is no need of assuming so much. The Parallel-Postulate follows from the existence of two unequal triangles of equal angle-sum, without regard to the proportionality of the sides or the respective equality of the angles.

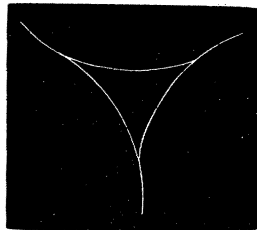
This remarkable truth was established in the very first Non-Euclidean Geometry published in 1733. After this date there existed in the world not only Euclid's, but also the two Non-Euclidean geometries; and so henceforth

the substitutes proposed for Euclid's postulatam lose interest, but I must still mention two, since they have come down to our very day.

One of these is a splendid illustration of the benefits of the study of non-Euclidean geometry, since though it deceived Sir Wm. Rowan Hamilton and John Casey, it would not now be accepted by the merest tyro who had looked into Lobatschewsky or Bolyai. It was not long ago exposed in *Nature* by that sound geometer O. Henrici, but I will quote its demolition by Perronet Thompson in 1833, who says of it: 'Professor Playfair in the Notes to his *'Elements of Geometry'* p. 409, has proposed another demonstration, founded on a remarkable *non causa pro causa*.

It purports to collect that (on the sides being successively prolonged to the same hand) the exterior angles of a rectilineal triangle are together equal to four right angles, from the circumstance that a straight line carried round the perimeter of a triangle by being applied to all the sides in succession, is brought into its old situation again; the argument being, that because this line has made the sort of somerset it would do by being turned through four right angles about a fixed point, the exterior angles of the triangle have necessarily been equal to four right angles.

The answer to which is, that there is no connexion between the things at all, and that the result will just as much take place where the exterior angles are avowedly not equal to four right angles. Take, for example, the plane triangle formed by three small arcs of the same or equal circles, as in the margin; and it is manifest that an arc of this circle may be carried round precisely in the way described and return to its old situation, and yet there be no pretense for inferring that the exterior angles were equal to four right angles.



And if it is urged [as actually was by John Casey] that these are *curved* lines and the statement made was of *straight*; then the answer is by demanding to know, what property of straight lines has been laid down or established, which determines that what is not true in the case of other lines is true in theirs.

It has been shown that, as a general proposition, the connexion between a line returning to its place and the exterior angles having been equal to four right angles, is a *non sequitur*; that it is a thing which may not be; that the notion that it returns to its place *because* the exterior angles have been equal to four right angles, is a mistake. From which it is a legitimate conclusion that if it has pleased nature to make the exterior angles of a triangle greater or less than four right angles, this would not have created the smallest impediment to the line's returning to its old situation after being carried round the sides; and consequently the line's returning is no evidence of the angles not being greater or less than four right angles."

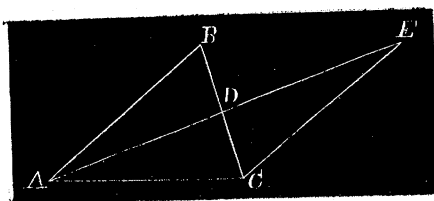
The other fallacy which I shall now proceed to mention occurred first in the Twelfth Edition of Legendre's geometry, but its viciousness having been

recognized by its author, it was afterwards withdrawn. It is, to say the least, queer, that this error, exploded and known as a fallacy for more than sixty years, should in 1892 have been reproduced by a paradoxer named J. N. Lyle, in a pamphlet printed by him in St. Louis.

Lobatschewsky's § 19 is (See Halsted's translation, 4th Ed. p. 16), *In a rectilineal triangle the sum of the three angles can not be greater than two right angles.*

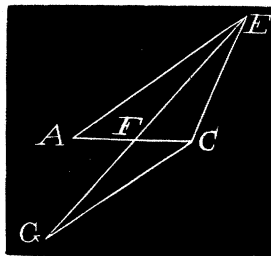
On the assumption that the straight is infinite, and that two straights which have crossed never recur or meet again, this is deduced by Lobatschewsky as follows:

Suppose in the triangle ABC the sum of the three angles is equal to $\pi + a$; then choose in case of the inequality of the sides the smallest BC , halve it in D , draw from A through D the line AD and make the prolongation of it, DE , equal to AD , then join the point E to the point C by the straight line EC .



In the congruent triangles ADB and CDE , the angle $ABD = DCE$, and $BAD = DEC$ (Theorems 6 and 10); whence follows that also in the triangle ACE the sum of the three angles must be equal to $\pi + a$.

But also the smallest angle BAC (Theorem 9) of the triangle ABC in passing over into the new triangle ACE has been cut up into the two parts EAC and AEC ; and the larger of the two angles ABC , BAC , namely ABC , has been taken away from the sum of these two, ABC and BAC , and added to the angle ACB . Now again in the new triangle ACE , halve the smallest side AC in F , draw from E through F the line EF and make the prolongation of it, FG , equal to EF , then join the point G to the point C by the straight line GC .



Thus we take away from the sum of the angles EAC and AEC the greater EAC and add it to the angle ACE .

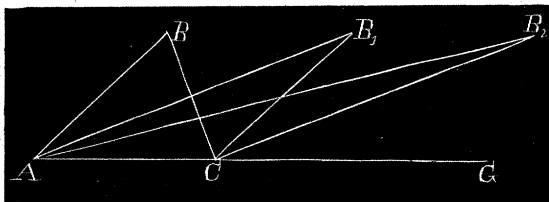
Continuing this process, continually halving the side opposite the smallest angle, we must finally attain to a triangle in which the sum of the three angles is $\pi + a$, but wherein are two angles, each of which in absolute magnitude is less than $\frac{1}{2}a$; since now however the third angle cannot be greater than π , so must a be either null or negative.

This demonstration by Lobatschewsky is in perfect accord with Euclid, since it is based upon the first proposition of Euclid's Tenth Book: *If from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and so on, there shall at length remain a magnitude less than the smaller of the proposed magnitudes.*

Here the smaller of the proposed magnitudes is $\frac{1}{2}a$, and the greater is the sum of ABC and BAC .

Now our paradoxer, Mr. Lyle, read this demonstration in my translation, and though, as we shall see, he did not understand it, yet he felt that because Lobatschewsky gave it, it must be rigorous, and so thought he might trust himself to reproduce it, but in doing so, laid bare in the most ludicrous manner, his utter failure to grasp it or to see its cogency.

He calls his blunder "Lobatschewsky's Theorem 19 recast in the rigidest (sic) Euclidian forms," and uses for his fallacious pseudo-proof this figure which at once tells us how he is about to lay bare his mental nakedness. Not understanding the proof, of course he does not comprehend that he must never halve the greater of two sides going to C , as AC , and CB_2 ; so as it happens he *always* halves the greater, thus repeating his blunder an indefinite number of times, and utterly vitiating a beautiful proof.



Instead of taking from the sum of ABC and BAC more than half and continually more than half of the remainder, he only takes a little piece that rapidly becomes less than a millionth part of their sum, while his seeming highway of demonstration dwindles to a squirrel-path and runs up a tree.

Nothing daunted, he reproduces on his next page the well-known fallacious demonstration from the twelfth edition of Legendre's geometry, that the sum of the three angles of a rectilinear triangle cannot be less than two right angles; only he makes ridiculous, by his "recast," what was always erroneous.

Lobatschewsky had this disgraceful fallacy of Legendre's before him when he wrote in 1800 the book I have translated, so I need only quote in regard to it a few sentences written in 1833 by Perronet Thompson. Referring to the above valid procedure as I have given it, he says: "the described process may be continued, till two of the angles of the last resulting triangle are together less than any magnitude that shall have been assigned; and consequently the third or remaining angle may be made to approach, within any magnitude however small it may be chosen to assign, to the sum of the three angles of the original or any of the intervening triangles. All this is irrefragable; but not so the proposition next taken for granted, which is that the third angle last mentioned approaches within any magnitude however small it may be chosen to assign, to the sum of two right angles" "The conclusion is founded on neglect of the very early mathematical truth, that continually increasing is no evidence of ever arriving at a magnitude assigned."